Comments on Mochizuki's 2018 Report

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These notes attempt to unravel some of Mochizuki's comments in his September 2018 *Report on discussions...*, which aims to support his claimed proof of the *abc* conjecture. I am not an arithmetic geometer or number theorist, but a category theorist, and these notes focus on category-theoretic issues and concepts which Mochizuki has raised. These notes make no claim as to the correctness or otherwise of Mochizuki's proof, or Scholze–Stix's rebuttal, but merely aim to extract concrete mathematical content from Mochizuki's *Report* in as clear terms as possible, and to examine Scholze–Stix's simplifications in light of this.

Mathematics is the art of giving the same name to different things
—Henri Poincaré

Background

In March 2018 Peter Scholze and Jacob Stix travelled to Japan to visit Shinichi Mochizuki to discuss with him his claimed proof of the *abc* conjecture. In documents released in September 2018, Scholze–Stix claimed the key Lemma 3.12 of Mochizuki's third *Inter-Universal Teichmüller Theory* (IUTT) paper reduced to a trivial inequality under certain harmless simplifications, invalidating the claimed proof.[2] Mochizuki agreed with the conclusion that under the given simplifications the result became trivial, but *not* that the simplifications were harmless. However, Scholze and Stix were not convinced by the arguments as to why their simplifications drastically altered the theory, and we stand at an impasse.

The impatient reader may wish to start at the *colimits and diagrams* example on page 3

[2] Scholze apparently had concerns about the proof of Lemma 3.12 for some time; it has been reported that a number of other arithmetic geometers independently arrived at the same conclusion.

The documents released by both sides[3] include two versions of a report by Scholze–Stix, titled *Why abc is still a conjecture*, each with an accomanying reply by Mochizuki, as well as a 41-page article, *Report on discussions, held during the period March 15 – 20, 2018, concerning Inter-Universal Teichmüller Theory (IUTCH)*. This latter document, which shall be be referred to as 'the *Report*', is written in a style consistent with Mochizuki's IUTT papers, and his other documents concerning IUTT. As such, it can be difficult (at least for me) to extract concrete and precisely-defined mathematical results that aren't mere analogies or metaphors. Rather than analogies, one should strive to express the necessary ideas or objections in as precise terms as possible, and I argue that one should use category theory to clean up all the parts of the arguments that are not actual number theory or arithmetic geometry.

[3] Avilable at http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html

Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial. —Peter Freyd

Category theory and structuralism

Not being an expert in number theory or arithmetic geometry, I shall focus on the category-theoretic aspects that Mochizuki invokes, and cast them in a way that makes the point clearer. I will also make reference to 'structural' thinking and reasoning. This is a trend that started in the mid-20th century with the work of Bourbaki and which was subsumed by category theory. One key idea is the *principle of equivalence*, taken to be fundamental by the Univalent Foundations of the late Vladimir Voevodsky: anything that can be said in mathematics should be invariant under isomorphism of the appropriate kind. Some objects are incredibly rigid, like the well-founded $\in$-trees that underlie the sets in ZFC, or the real numbers as a complete ordered field; neither of these have nontrivial automorphisms. On the other hand, some objects have very large symmetry groups, like sets in the category of sets and functions, or the real numbers as a metric space. As should be clear from the preceding examples, the *context* in which one considers mathematical objects is absolutely crucial: one cannot discuss the group of automorphisms of a mathematical object without specifying in which category it lives.

For background on category theory, see the classic *Categories for the working mathematician* (Springer), by Mac Lane, or the more recent *Category theory in context* (Dover), by Riehl, available from `http://www.math.jhu.edu/~eriehl/context/`, or *Conceptual Mathematics* (Cambridge University Press), by Lawvere and Schanuel, or *Category theory for the sciences* (MIT Press), by Spivak, free draft version available from `http://math.mit.edu/~dspivak/CT4S.pdf`

Another key point is that one can substitute one object X for another isomorphic object X', provided one has a *specified isomorphism* $b\colon X \xrightarrow{\sim} X'$ (for *b*ase) between them. If one is given another isomorphism $c\colon X \xrightarrow{\sim} X'$ (or *c*omparison), then one can compare these isomorphisms to check if they are equal or not. If one is identifying X and X', this amounts to checking whether $\mathrm{id}_{X'} = b \circ b^{-1}$ and $c \circ b^{-1}$ are equal. In particular, one cannot assume that c corresponds to the identity map on X'. Another way to think about this is that the set $\mathrm{Isom}(X, X')$ of isomorphisms has a free and transitive left $\mathrm{Aut}(X')$ action, and that choosing a basepoint $b \in \mathrm{Isom}(X, X')$ is equivalent to choosing a bijection of left $\mathrm{Aut}(X')$-spaces $\mathrm{Aut}(X') \xrightarrow{\sim} \mathrm{Isom}(X, X')$. In particular, one should only think of $\mathrm{Aut}(X')$ as a left $\mathrm{Aut}(X')$-space, not a group.

This example is meant to illustrate an abstract category-theoretic principle: isomorphic objects are fungible, as long as the isomorphism is consistently incorporated in the substitution. Moreover, the category or categories that one is working with constitute crucial information in this process, and functoriality makes things precise that otherwise can be vague. One is also forced to be explicit when arbitrary choices have been made, and the dependencies on these choices can be analysed—in the best situations different choices lead to uniquely isomorphic results, at which point one can know that different choices do not make a difference to the resulting mathematics.

This idea seems to be at the heart of Mochizuki's rebuttal (and also seems to be a key part of IUTT in general), but not expressed in a way that a category theorist would phrase it. It is rather conveyed in a wordy way that doesn't specify cleanly what categories are being used, when objects have in fact had forgetful functors applied to them, or when one is using a lift of an object through a forgetful functor.

That said, there are a number of things that Mochizuki writes that feel to me like vestiges of 'material' thinking. Here by *material* thinking[4] I mean, in opposition to structural thinking, the platonic attitude that specific representations of objects matter and make a difference to the mathematics. For instance, and this is mentioned and emphasised repeatedly, Mochizuki insists on special distinct labels be applied to copies of the same object and claims this makes a real difference to results. This point is a key point of contention with Scholze–Stix, who claim that their omission of such labels do not affect the result. At this point in time, I cannot tell whether Scholze–Stix's simplifications preserve structural information (their claim) or lose structural information (Mochizuki's claim). These notes are to rather going to examine Mochizuki's examples in the *Report* to find precise mathematical statements underlying them, and what thinking about them structurally can say. At times Mochizuki's examples are formulated so that the structural content is unclear (mostly because the required categories are not supplied), and at times they are expressed in a material way, but with an underlying structural idea obscured by the jargon.

[4] For a more precise version see Michael Shulman's paper *Comparing material and structural set theories*, `arXiv:1808.05204`

I will start by considering an abstract category-theoretic setup that underlies several of Mochizuki's examples, shorn of all irrelevant information and commentary.

Example: colimits and diagrams

For this example, fix a category C with at least countable colimits. Let L be a countable set equipped with an unbounded, nowhere dense linear order, considered as a small category, and consider diagrams[5] $X\colon L \to C$. These diagrams consist of bi-directional sequences of objects X_ℓ linked by morphisms $t_{\ell\,\ell'}\colon X_\ell \to X_{\ell'}$ with $t_{\ell\,\ell''} = t_{\ell'\,\ell''} \circ t_{\ell\,\ell'}$ and $t_{\ell\,\ell} = \mathrm{id}_{X_\ell}$. The colimit of such a diagram is an object X_∞ of C equipped with morphisms $c_\ell\colon X_\ell \to X_\infty$ such that $c_{\ell'} \circ t_{\ell\,\ell'} = c_\ell$ (the data $(X_\infty, \{c_\ell\})$ is called a *cocone*) satisfying the required universal property. Given a second universal cocone $(X'_\infty, \{c'_\ell\})$ there is a unique isomorphism $u\colon X_\infty \xrightarrow{\sim} X'_\infty$ such that $c'_\ell = u \circ c_\ell$. Any one of the objects X_∞ can be called $\operatorname{colim} L$, but it implicitly comes equipped with the rest of the cocone data. One way to construct such a colimit and cocone data is to take a certain quotient of the coproduct $\coprod_{\ell\in\mathrm{Obj}(L)} X_\ell$, the quotient being by the smallest equivalence relation that forces the equations $c_{\ell'} \circ t_{\ell\,\ell'} = c_\ell$ to hold.

[5] Such diagrams are considered by Mochizuki in (LbEx2) and (LbEx3) in the *Report*, albeit using $\mathbb{Z}$ instead of L; here I am avoiding any hint of chosen element 0.

Now it happens that the $\operatorname{colim} L$ can be calculated using completely different non-isomorphic diagrams. For instance, taking any subset $R \subset L$ with the property that for all $\ell \in L$, there is some $r \in R$ with $\ell \leq r$, one finds that the unique induced map

$\operatorname{colim} R \to \operatorname{colim} L$ is an isomorphism. One might make this situation more concrete by taking $L = \mathbb{Z}$ and $R = \mathbb{N}$, but these choices are immaterial; all one requires is that L is equipped with a 'successor' map $s\colon L \to L$ such that $\ell < s(\ell)$.

Now we come to the specific sticking point Mochizuki seems to be addressing. Consider now the case that all X_ℓ are the same object A, and all the morphisms $t_{\ell\, s(\ell)}$ are the same endomorphism t. As before, one can take a suitable $R \subset L$ to calculate this, now taken to be closed under s, and consider the restricted diagram $X_R\colon R \hookrightarrow L \xrightarrow{X} C$. The assumptions on R make it uniquely isomorphic to the additive monoid $\mathbb{N}$, and which defines a one-object category $\mathbf{B}\mathbb{N}$. The R-shaped diagram X_R in C factors as

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, "X_R"] & \mathbf{B}\mathbb{N} \arrow[d, "\overline{X}"] \\ L \arrow[r, "X"'] & C \end{tikzcd}$$

where the right vertical functor picks out the object A and sends $n \mapsto t^n$ (with 0, corresponding to the minimum element of R, being sent to id_A). The point that Mochizuki seems to be making in (LbEx2) and (LbEx3) in the *Report* is that while $\operatorname{colim} X \simeq \operatorname{colim} X_R$,

$$\operatorname{colim} X_R \not\simeq \operatorname{colim} \overline{X}.$$

Notice that both diagrams have the same *image* in C, but domain of the diagram makes a huge difference.[6]

[6] For example, if $C = \mathsf{Set}$, $A = \mathbb{N}$, then $\operatorname{colim} \overline{X} = \{0\}$ but $\operatorname{colim} X_R = \mathbb{Z}$.

This idea of the previous paragraph takes Mochizuki a page of text[7] to explain and includes reference to "confusion" and "internal contradiction"s around the "erroneous operation" of "omitting the labels". To treat the construction properly, one is not creating labels, or seperate additional copies of objects, but merely taking the definition of colimit seriously. In particular, one is not ignoring the functorial nature of the colimit diagram, even if its image consists of a single object and iterations of a single endomorphism.

[7] My treatment here is only so long so as to not presume the technical definition of colimit.

An even more striking example is provided by abstracting the example (LbEx5) of the *Report*. Instead of a diagram of shape L or R, consider instead the discrete category $\mathrm{disc}(I)$ specified by a (countable) set I, with only identity arrows. Then the colimit of a diagram $\mathrm{disc}(I) \to C$ where every $i \in I$ is mapped to the same object A of C is the I-fold coproduct $\coprod_I A$ of copies of A, or equivalently, the copower $I \pitchfork A$. Now of course one has the factorisation $\mathrm{disc}(I) \to * \xrightarrow{A} C$, and the colimit over a trivial diagram $* \xrightarrow{A} C$ is just the object A again, so keeping the information of the diagram shape is crucial. To join up with (LbEx5), every manifold is the quotient of some coproduct $\coprod_I \mathbb{R}^n$ in the category of manifolds, by an equivalence relation that is itself some other coproduct $\coprod_J \mathbb{R}^n$. Nothing here reduces to the triviality that (LbEx5) seems to claim, despite all diagrams only ever using a single copy of $\mathbb{R}^n$.

Another example, Mochizuki's (LbEx4), deals with subfields of $\mathbb{C}$. Note that one can consider pushout diagrams in C of shape

$S := (1 \leftarrow 0 \rightarrow 1')$ that factor (up to isomorphism) through $S \to P := (0 \rightrightarrows 1)$, after applying a forgetful functor[8] $U\colon C \to C'$. Let us further assume that U is faithful, and even injective on objects. We thus have a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & C \arrow[d, "U"] \\ P \arrow[r] \arrow[ur, phantom, "\overset{\simeq}{\swArrow}"] & C' \end{tikzcd}$$

but if U does not preserve enough colimits, then while the image of the diagram of shape S in C' consists entirely of objects and arrows in the image of U, then any colimit of $S \to C'$ (if it exists) may not be the same as the colimit in C. In the example (LbEx4), C has at most one arrow between any two objects, so is a poset, so the colimit of a diagram is merely the supremum of the corresponding subset of the partially ordered set determined by C. The poset of subfields of $\mathbb{C}$ has suprema, and so considered as a category it has colimits, in particular pushouts, but the category of fields does not. Mochizuki describes the pushout diagram as

> "... a situation in which one has **two distinct abstract fields** F_1, F_2 that are **glued together along the subfields** $\mathbb{Q} \subseteq F_1$, $\mathbb{Q} \subseteq F_2$".

In this sentence it seems that this diagram is being considered in C' rather than C (i.e. abstract fields, not subfields of $\mathbb{C}$). It is indeed possible to consider the pushout diagram as giving rise to a formal colimit, for instance in the free cocompletion. In the free cocompletion one does not mind that the two fields F_1 and F_2 are isomorphic, and one could in fact replace the pushout diagram by one that factors through P, as above, without affecting the answer. However, the *free* cocompletion[9] ignores any existing colimits, and they cease being colimits. Worse, this discussion is happening in C', and the colimit in C (namely the field K, in Mochizuki's notation) is not even sent to a colimit by the forgetful functor—and in any case, F_1 and F_2 are *not* isomorphic in C. So it transpires that U is not conservative, and the pushout diagram does not factor through P in C. It is thus unclear that the various constructions being discussed in (LbEx4) are even in the same category, and the functor relating the two obvious candidate categories is not well-behaved enough to preserve them.

Mochizuki discusses 'labels' a lot, but it appears what is really meant is that for the purposes of considering (formal) colimits, one needs to not discard the domain of the diagram. It may well be that Mochizuki's intention is to capture this idea, but his mode of expressing such a simple category-theoretic construction obscures its simplicity. One can look at (H1) and (H2) in the *Report* for instance, and wonder what 'histories of operations' is supposed to mean, or 're-initialization operations'. If the diagrams shown there are supposed to represent diagram shapes over which one is taking colimits, then it is a category-theoretic triviality that one gets different colimits (recall the quote of Freyd above!).

[8] Mochizuki's example (LbEx4) has $C = \mathsf{SubFields}(\mathbb{C})$, $C' = \mathsf{Fields}$.

[9] Note that if one considers fields as objects of the category of rings, then there *is* a pushout, namely the tensor product $F_1 \otimes F_2$, which is then not a field.

Example: polymorphisms

Another sticking point in the discussions was that Mochizuki insists that 'polymorphisms' are an essential structure to consider, whereas Scholze–Stix were not convinced of the necessity. If one turns to IUTT1§0, one finds that a polymorphism in a category C from an object X to an object Y is a subset of $C(X,Y)$. Of particular importance seems to be poly-isomorphisms, which are subsets of $\mathrm{Isom}(X,Y)$, and even 'full' poly-isomorphisms, which are the set $\mathrm{Isom}(X,Y)$ itself. Two examples are given:

- Trival polymorphisms, which correspond to the set of those maps $X \to Y$ in C that lift some given map $X/_{\sim} \to Y/_{\approx}$ between the quotients;
- Nontrivial polymorphisms, key examples of which correspond to equivalence classes of maps in C that lift a given map in a quotient category $C/_{\sim}$ (for instance CW-complexes with homotopy classes of continuous maps).

If one is working with polymorphisms in the second sense, then one is not working in the category C, but with the category $C/_{\sim}$. Similar remarks can be made if one is only working with poly-isomorphisms. I find it curious that Mochizuki wants to work with full poly-isomorphisms[10], which in the second example correspond to descending to the groupoid where objects are uniquely isomorphic. It may be that this is a misreading of the situation, since I do not understand the complex web of data to which Mochizuki applies these ideas.

One thing that might be happening is that Mochizuki actually uses the subset $P \subseteq Isom(X,Y)$ as an object of interest itself, particularly if it is defined as being an orbit of some naturally acting group of automorphisms[11]. The group action, might indicate that one wants to think of a quotient category, but it is unclear. A clean category-theoretic treatment of what is going on would better define the rôle of polymorphisms.

[10] "This full poly-isomorphism [part of the data of the Θ-link] gives rise to the indeterminacies (Ind1, 2), which play a central role in IUTch.", *Report* §6.

[11] See for instance (VUC3) in the *Report*.

The Scholze–Stix simplifications

So given all this discussion of peculiarities on Mochizuki's side, what can be said about the approach of Scholze–Stix? Many times they say they are identifying certain objects of interest that are known to be isomorphic/equivalent. Mochizuki objects to this, but it is not a priori clear that identifying objects is destructive: in the examples above of colimits, one did not need to ensure that different objects were the values of different nodes in the diagram shape. The book-keeping is taking place at the diagram level, not at the specific identity of the objects.

However, one can go too far in this process. Recalling the discussion in the section 'Category theory and structuralism' above, one may identify objects X or X' assuming one has a given isomorphism between them, or else choosing a specified isomorphism $b\colon X \xrightarrow{\sim} X'$. If one then has some *other* isomorphism, then it can be turned into an automorphism of X' (say). Consider the case one has some diagram[12] $X\colon D \to C$ of objects where all the objects $X(d)$ in the image of the diagram are known to be isomorphic to a fixed object X_0. Then given an isomorphic diagram $X'\colon D \to C$, via some given natural isomorphism $\alpha\colon X \overset{\sim}{\Rightarrow} X'$, and where $X'(d) = X_0$ for all $d \in D$, there is a canonical isomorphism $\operatorname{colim} X \simeq \operatorname{colim} X'$. There is no guarantee[13] that the arrows of D are sent to identity maps by X'; in fact if the arrows in the image of X are not invertible, then neither will the arrows in the image of X'. What is going on is that even though one might assume for simplicity that all the objects of the diagram are sent to the same object, assuming that all the arrows in the diagram between them are identity arrows may be an obstruction to the existence of the natural isomorphism α, and hence to the existence of an isomorphism between the (formal) colimits.

[12] I am overloading notation purposely here, since Mochizuki seems to want to think of diagrams as stand-ins for formal colimits

[13] There is however a special case in which it *is* permissible to assume all the arrows in the image of X' are identity maps, and that is when the diagram X factors up to isomorphism through the trivial category $*$.

Another tactic that Scholze–Stix use is looking at diagrams transferred through some equivalence $E\colon C \to C'$ of categories[14]. This is particularly useful if the objects and arrows of C' are a lot simpler to describe, and it may even be the case that C' has all objects isomorphic, even if there are many non-invertible maps. Note that equivalences of categories commute with colimits, and the free cocompletions of equivalent categories are equivalent, so one is free to consider diagrams in a one-object category C' as giving elements of the free cocompletion of C. Again, I emphasise that diagrams $D \to C'$, where C' is a one-object category, can give rise to nontrivial results in the free cocompletion of C'. There is no mathematical reason why calculations cannot proceed in this manner wherever possible.

[14] For instance, they quote Mochizuki's Theorem (Theorem 7 in their report); see also §2.1.4.

Conclusion

These notes have attempted to cast some of the examples proposed by Mochizuki to answer to Scholze–Stix in a more category-theoretic light. Ideally all discussions about the content of IUTT can be addressed in such precise terms, rather than worry about things like

> "the risk that **different people** will "remember" **different labeling appartuses**, which result in **structurally non-equivalent mathematical structures**", *Report* (DfLb)

By replacing discussion of psychology and suggestive metaphors by rigorous definitions of all the categories in which objects live, and keeping track of forgetful functors, communication about IUTT can focus on the difficult mathematical content, rather than about whether or not objects need specific labels.

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